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Colength sequences for matrices

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In this paper we study the cocharacter sequence for $k \times k$ matrices over a field of characteristic zero. Let χ_n be the nth cocharacter. Then χ_n is a character for the symmetric group S_n and it can be decomposed as a sum of irreducible S_n-characters,

$$\chi_n = \sum_{\lambda \in \mathrm{Par}(n)} m_\lambda \chi^\lambda.$$

The determination of the multiplicities remains one of the basic unsolved problems in the study of the identities and invariants of matrices. We define $l(n)$, the nth colength to be $l(n) = \sum m_\lambda$. The main theorem of this paper is that

$$l(n) = \alpha n^{\binom{k^2}{2}} + O\left(n^{\binom{k^2}{2}-1}\right), \tag{1}$$

for some constant α. In fact, we will prove more. There are three more cocharacter sequences associated with $k \times k$ matrices: the pure trace cocharacter sequence, the mixed trace cocharacter sequence, and the cocharacter sequence of the center. We will prove that each of these cocharacter sequences has length $\alpha n^{\binom{k^2}{2}} + O(n^{\binom{k^2}{2}-1})$, for some (possibly different) α.

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1. The lower bound

Our proofs will principally use Poincaré series instead of S_n-characters. For the equivalence of the two methods, see [1] and [5]. Let R be the algebra generated by k^2 generic $k \times k$ matrices, let C be the center of R, let $\overline{C}$ be the algebra generated by the traces of elements of R, and let $\overline{R}$ be the algebra generated by R and $\overline{C}$. Each of these algebras has a k^2-fold grading and so each has a Poincaré series in the k^2 variables $x_1, \ldots, x_{k^2}$. Moreover, each is a symmetric function, and so can be written as a series in the Schur functions. We will use the symbol m_λ, or $m_\lambda(-)$ to denote the multiplicity of the Schur function on the partition λ. So, for example,

$$P(R) = \sum_{n=0}^{\infty} \sum_{\lambda \in \mathrm{Par}(n)} m_\lambda(R) S_\lambda(x_1, \ldots, x_{k^2}).$$

It is known that the multiplicities $m_\lambda(-)$ of the Schur functions in the Poincaré series are also equal to the multiplicities of the irreducible S_n-character χ^λ in the corresponding cocharacter sequence. Again, we let $l(-)(n)$ be the sum of the multiplicities m_λ, where λ runs over the partitions of n.

Lemma 1. *In order to prove Eq.* (1) *for R and C it suffices to prove it for $\overline{R}$ and $\overline{C}$.*

Proof. It follows from the work of Formanek in [5] and [6] that the multiplicities for generic matrices with and without trace are closely related. It is easy to see that each $m_\lambda(R) \leqslant m_\lambda(\overline{R})$ and $m_\lambda(C) \leqslant m_\lambda(\overline{C})$. Moreover, let μ be the partition of k^2 with all parts equal to one, $\mu = (1, \ldots, 1)$. Then, for any partition λ, and any positive integer d, Formanek proved that

$$\begin{aligned} m_\lambda(\overline{R}) &= m_{\lambda+d\mu}(\overline{R}), \qquad & m_{\lambda+2\mu}(R) &= m_{\lambda+2\mu}(\overline{R}), \\ m_\lambda(\overline{C}) &= m_{\lambda+d\mu}(\overline{C}), \qquad & m_{\lambda+2\mu}(C) &= m_{\lambda+2\mu}(\overline{C}). \end{aligned}$$

It follows from this that

$$\begin{aligned} l(R)(n) &= \sum_{\lambda \in \mathrm{Par}(n)} m_\lambda(R) \geqslant \sum_{\lambda \in \mathrm{Par}(n-2k^2)} m_{\lambda+2\mu}(R) \\ &= \sum_{\lambda \in \mathrm{Par}(n-2k^2)} m_{\lambda+2\mu}(\overline{R}) = \sum_{\lambda \in \mathrm{Par}(n-2k^2)} m_\lambda(\overline{R}) \end{aligned} \tag{2}$$

and likewise for C and $\overline{C}$. Hence,

$$l(\overline{R})(n-2k^2) \leqslant l(R)(n) \leqslant l(\overline{R})(n) \quad \text{and} \tag{3}$$

$$l(\overline{C})(n-2k^2) \leqslant l(C)(n) \leqslant l(\overline{C})(n). \tag{4}$$

This shows that it suffices to prove our bounds for $l(\overline{C})(n)$ and $l(\overline{R})(n)$, and the corresponding bounds for $l(C)(n)$ and $l(R)(n)$ will follow. □

We now record a somewhat technical lemma from [2]. Let R^0 be the algebra generated by k^2 generic trace zero matrices, C^0 the center of R^0, etc. Then Theorems 4 and 5 of that work compare the Poincaré series of $\overline{C}$ with that of $\overline{C}^0$, and the Poincaré series of $\overline{R}$ with that of $\overline{R}^0$.

Lemma 2. *With notation as above,*

$$P(\overline{C}) = \prod_i (1-t_i) P(\overline{C}^0) \quad \textit{and} \quad P(\overline{R}) = \prod_i (1-t_i) P(\overline{R}^0).$$

Moreover, if the Poincaré series of $\overline{C}^0$ and $\overline{R}^0$ are expanded in terms of Schur functions, then all of the multiplicities are non-negative.

In terms of S_n-characters, Lemma 2 is equivalent to

$$\chi_n(\overline{C}) = \sum_{i=0}^{n} \chi^{(i)} \hat{\otimes} \chi_{n-i}(\overline{C}^0), \tag{5}$$

$$\chi_n(\overline{R}) = \sum_{i=0}^{n} \chi^{(i)} \hat{\otimes} \chi_{n-i}(\overline{R}^0), \tag{6}$$

where the tensor is the outer tensor product and may be calculated using Young's rule, see [8]. Young's rule computes the outer tensor product of an irreducible character χ^λ with an irreducible character corresponding to a one part partition $\chi^{(i)}$. It says that

$$\chi^\lambda \hat{\otimes} \chi^{(i)} = \sum \{\chi^\mu \mid \lambda \subset \mu,\ |\mu/\lambda| = i,\ \text{and for each } s,\ \lambda_s \leqslant \mu_s \leqslant \lambda_{s+1}\}.$$

It follows easily that if χ^λ is an irreducible S_n-character and if $i < j$, then the length of $\chi^\lambda \hat{\otimes} \chi^{(i)}$ is less than or equal to the length of $\chi^\lambda \hat{\otimes} \chi^{(j)}$. Hence, Eqs. (5) and (6) imply the following corollary.

Corollary 3. *Each of the colength sequences $l(\overline{C})(n)$ and $l(\overline{R})(n)$ is non-decreasing.*

In the introduction we defined a cocharacter sequence and a colength sequence. In this section we need to make use of a third sequence, the (homogeneous) codimension sequence. It is defined by

$$d_n(-) = \sum_{\lambda \in \mathrm{Par}(n)} m_\lambda(-) S_\lambda \underbrace{(1, \ldots, 1)}_{k^2}. \tag{7}$$

Note that $d_n(-)$ is the dimension of the part of the algebra in question (namely, C, R, $\overline{C}$ or $\overline{R}$) with total degree n. We will use the structure theory of $\overline{C}$ to get a lower bound on $d_n(\overline{C})$. In order to combine it with (7) and get a bound on $l(\overline{C})(n)$ we will need an estimate on the $S_\lambda(1,\dots,1)$. Note that $S_\lambda(1,\dots,1)$ is the dimension of the irreducible $\mathrm{GL}(h)$-module corresponding to the partition λ.

Lemma 4. *For $\lambda = (\lambda_1,\dots,\lambda_h)$ a partition of n with at most h non-zero parts,*

$$S_\lambda \underbrace{(1,\dots,1)}_{h} \leqslant (n+1)^{\binom{h}{2}}.$$

Proof. Let $d(\lambda,h)$ be $S_\lambda(1,\dots,1)$, with h 1's. It follows from Young's rule, see [8], that

$$d(\lambda,h) = \sum_{\mu_1=\lambda_2}^{\lambda_1} \sum_{\mu_2=\lambda_3}^{\lambda_2} \cdots \sum_{\mu_{h-1}=\lambda_h}^{\lambda_{h-1}} d(\mu,h-1).$$

By induction, $d(\mu,h-1) < (|\mu|+1)^{\binom{h-1}{2}} < (n+1)^{\binom{h-1}{2}}$. The number of terms in the sum is less than $(n+1)^{h-1}$. Hence

$$d(\lambda,h) < (n+1)^{h-1}(n+1)^{\binom{h-1}{2}} = (n+1)^{\binom{h}{2}}. \quad \square$$

Here is the lower bound on d_n.

Lemma 5. *There is a polynomial $f(n)$ of degree $2\binom{k^2}{2}$ such that $d_n(\overline{R}) > d_n(\overline{C}) > f(n)$ for all large enough n.*

Proof. Procesi proved that the quotient field of $\overline{C}$ is of transcendence degree $(k^2-1)k^2+1$ over its center, see [7]. It follows that $\overline{C}$ contains $(k^2-1)k^2+1$ homogeneous, algebraically independent elements. Let the degrees be $a_1,\dots,a_{(k^2-1)k^2+1}$, and let K be the subalgebra they generate. Then the dimension of the nth homogeneous part of K is the number of non-negative integer solutions to the equation

$$a_1x_1 + \cdots + a_{(k^2-1)k^2+1}x_{(k^2-1)k^2+1} = n.$$

This dimension is approximately $\alpha n(k^2-1)k^2$ if $\ell = \gcd\{a_i\}$ divides n, and is zero otherwise. This implies that $d_n(\overline{C}) > f(n)$ for n a large enough multiple of ℓ.

To get a bound for the other values of n (if there are any), we claim that $d_n(\overline{C})$ is an increasing function. This is because multiplication by $\mathrm{tr}(x_1)$ is a one-to-one map from $\overline{C}$ to itself that increases degree by 1. Hence, for any n, let $m = \ell\lfloor n/\ell\rfloor$. Then

$$d_n > d_m > f(m) > f(n-\ell). \quad \square$$

Here, now is a lower bound for the colengths.

Lemma 6. *There exists a polynomial $f(n)$ of degree $\binom{k^2}{2}$ such that $l_n(\overline{R}), l_n(\overline{C}) > f(n)$ for all large n.*

Proof. By Eq. (7),

$$d_n(\overline{C}) = \sum_{\lambda\in\mathrm{Par}(n)} m_\lambda S_\lambda \underbrace{(1,\dots,1)}_{k^2} \leqslant \max\{S_\lambda(1,\dots,1)\} \sum_\lambda m_\lambda.$$

Now apply Lemmas 4 and 5:

$$\alpha n^{k^2(k^2-1)} < (n+1)^{\binom{k^2}{2}} l(C)(n).$$

The case of $l_n(\overline{C})$ follows, and the case of $l_n(\overline{R})$ is similar. □

2. The upper bound

There is an inner product on symmetric functions in k variables given by

$$\langle f, g\rangle = \frac{1}{k!}(2\pi i)^{-k} \oint_T f(z_1,\dots,z_k) g\left(z_1^{-1},\dots,z_k^{-1}\right) \prod_{i\neq j}\left(1-\frac{z_i}{z_j}\right) \mathrm{d}\nu, \tag{8}$$

where T represents the torus $|z_i| = 1$ for all i, and $\mathrm{d}\nu$ is the measure

$$\mathrm{d}\nu = \frac{\mathrm{d}z_1}{z_1} \wedge \cdots \wedge \frac{\mathrm{d}z_k}{z_k}.$$

Using this inner product, Formanek derived a formula for the multiplicities $m_\lambda(\overline{C})$ and $m_\lambda(\overline{R})$ in [5, Theorem 12].

Formanek's Theorem. *The multiplicities $m_\lambda(\overline{C})$ and $m_\lambda(\overline{R})$ are given by*

1. $m_\lambda(\overline{C}) = \langle S_\lambda(\frac{z_i}{z_j}), 1\rangle$.
2. $m_\lambda(\overline{R}) = \langle \sum_{i,j=1}^k \frac{z_i}{z_j} S_\lambda(\frac{z_i}{z_j}), 1\rangle$.

We define $L(\overline{C}, t)$, $L(\overline{R}, t)$ to be the generating functions $\sum l(\overline{C})(n)t^n$ and $\sum l(\overline{R})(n)t^n$, respectively. Using the identity $S_\lambda(x_1t, x_2t, \dots) = t^{|\lambda|} S_\lambda(x_1, x_2, \dots)$ we get

$$L(\overline{C}, t) = \sum_n \sum_{\lambda\in\mathrm{Par}(n)} \left\langle S_\lambda\left(\frac{z_i}{z_j}\right), 1\right\rangle t^n = \left\langle \sum_\lambda S_\lambda\left(\frac{z_i}{z_j}t\right), 1\right\rangle \quad \text{and}$$

$$L(\overline{R}, t) = \sum_n \sum_{\lambda\in\mathrm{Par}(n)} \left\langle \sum_{i,j=1}^k \frac{z_i}{z_j} S_\lambda\left(\frac{z_i}{z_j}\right), 1\right\rangle t^n = \left\langle \sum_\lambda \sum_{i,j=1}^k \frac{z_i}{z_j} S_\lambda\left(\frac{z_i}{z_j}t\right), 1\right\rangle.$$

We now apply to this the following theorem of Littlewood, see [9, I,5 Ex. 4]:

$$\sum_{\lambda} S_\lambda(x_1,\dots,x_n)=\prod_i(1-x_i)^{-1}\prod_{i<j}(1-x_ix_j)^{-1}.$$

Combining Formanek's theorem with Littlewood's identity yields the next lemma.

Lemma 7. *The series $L(\overline{C},t)$ and $L(\overline{R},t)$ may be calculated by*

1. $L(\overline{C},t)=\langle\prod_{i,j=1}^k(1-\frac{z_i}{z_j}t)^{-1}\prod_{(a,b)<(c,d)}(1-\frac{z_a}{z_b}\frac{z_c}{z_d}t^2)^{-1},1\rangle$.
2. $L(\overline{R},t)=\langle\sum_{i,j=1}^k(\frac{z_i}{z_j})^2\prod_{i,j=1}^k(1-\frac{z_i}{z_j}t)^{-1}\prod_{(a,b)<(c,d)}(1-\frac{z_a}{z_b}\frac{z_c}{z_d}t^2)^{-1},1\rangle$.

Here the second product in each formula uses any total order (*such as the lexicographical one*) *on the ordered pairs.*

The rest of this section will be devoted to the study of $L(\overline{C},t)$ and $L(\overline{R},t)$ using complex integrals. We note that $L(\overline{C},t)$ is, up to a constant factor, the integral over T of the following function:

$$\frac{\prod_{i\neq j}(1-\frac{z_i}{z_j})}{\prod_{i,j}(1-\frac{z_i}{z_j}t)\prod_{(a,b)<(c,d)}(1-\frac{z_a}{z_b}\frac{z_c}{z_d}t^2)}\tag{9}$$

and that $L(\overline{R},t)$ is, up to a constant factor, the integral over T of:

$$\frac{\sum_{i,j}z_iz_j^{-1}\prod_{i\neq j}(1-\frac{z_i}{z_j})}{\prod_{i,j}(1-\frac{z_i}{z_j}t)\prod_{(a,b)<(c,d)}(1-\frac{z_a}{z_b}\frac{z_c}{z_d}t^2)}.\tag{10}$$

It follows from Cauchy's theorem, as we will show below, that each $L(t)$ is a rational function of t in which all poles are at roots of 1. Consider the partial fraction decompositions:

$$L\big(\overline{C},t\big)=p(t)+\sum\frac{c(\omega,a)}{(\omega-z)^a},\qquad L\big(\overline{R},t\big)=q(t)+\sum\frac{r(\omega,a)}{(\omega-z)^a}\tag{11}$$

where $p(t)$ and $q(t)$ are polynomials and where the ω are roots of 1. Note that

$$(\omega-z)^{-a}=\omega^{-a}(1-z/\omega)^{-a}=\omega^{-a}\sum\binom{n+a}{a-1}(z/\omega)^n.$$

Hence, for n greater than the degrees of $p(t)$ and $q(t)$,

$$l(\overline{C})(n) = \sum_{\omega,a} c(\omega, a)\omega^{-n-a}\binom{n+a}{a-1}, \tag{12}$$

$$l(\overline{R})(n) = \sum_{\omega,a} r(\omega, a)\omega^{-n-a}\binom{n+a}{a-1}. \tag{13}$$

We now apply Corollary 3 to show that the pole of highest order occurs at $t = 1$.

Lemma 8. *The functions $L(\overline{C}, t)$ and $L(\overline{R}, t)$ each have a unique pole of highest order at $t = 1$.*

Proof. For $L(t) = L(\overline{C}, t)$ or $L(\overline{R}, t)$ let the poles of highest order be at $\omega_1, \ldots, \omega_k$ which are all roots of 1. Then $l(n)$ is asymptotic to

$$\alpha_1\omega_1^n n^d + \cdots + \alpha_k\omega_k^n n^d$$

where the poles are of order $d + 1$. Let

$$c(n) = \alpha_1\omega_1^n + \cdots + \alpha_k\omega_k^n,$$

so $l(n)$ is asymptotic to $c(n)n^d$. Since the $l(n)$ are real and non-decreasing, the $c(n)$ must be real and non-decreasing for large n. However, since the ω are roots of 1, the $c(n)$ are periodic. This implies that the sequence $c(n)$ is constant and so the only root of 1 that occurs is 1 itself. $\square$

We now need to bound the degree of the pole at $t = 1$ in each $L(t)$. In the denominator of (9) and of (10) the product $\prod_{i,j}(1 - \frac{z_i}{z_j}t)$ contains k factors of $(1 - t)$. Pulling them out and including the factor of $\frac{1}{z_1 \cdots z_k}$ from the measure, then $L(\overline{C}, t)$ is the integral of

$$I_k = \prod_{i \neq j} \frac{(1 - \frac{z_i}{z_j})}{(1 - \frac{z_i}{z_j}t)} \prod_{(a,b)<(c,d)} \frac{1}{(1 - \frac{z_a}{z_b}\frac{z_c}{z_d}t^2)} z_1^{-1} \cdots z_k^{-1}.$$

For $L(\overline{R}, t)$, we also expand the sum $\sum z_i z_j^{-1}$ in the numerator to get I_k a sum of integrals of the form

$$\prod_{i \neq j} \frac{(1 - \frac{z_i}{z_j})}{(1 - \frac{z_i}{z_j}t)} \prod_{(a,b)<(c,d)} \frac{1}{(1 - \frac{z_a}{z_b}\frac{z_c}{z_d}t^2)} z_1^{\alpha_1} \cdots z_k^{\alpha_k}$$

where the exponents α_i sum to $-k$.

Lemma 9. *$L(\overline{C}, t)$ and $L(\overline{R}, t)$ are rational functions of t. Each can be written with denominator a product of terms of the form $(1 - t^a)$. The order of the pole at $t = 1$ is at most $\binom{k^2}{2} + 1$.*

Proof. We inductively define I_a for $a = 0, \dots, k$. We start with I_k as above and our goal is to have $(1-t)^{-k} I_0$ equal to $L(t)$. For the intermediate values of a, each I_a will be a sum of terms, each an a-fold integral, representing one term in the integration of I_{a+1}.

By Cauchy's residue theorem, we can perform an integral $\oint f\,\mathrm{d}z$ about the unit circle by summing the residues inside the circle. Given a pole at $z = u$ of order N, the residue at u is

$$\lim_{z \to u} \frac{\mathrm{d}^{N-1}}{\mathrm{d}z^{N-1}} \bigl((z-u)^N f\bigr). \tag{14}$$

Now, I_k can be written in the form

$$f(t) \prod_{i=1}^{I} \left(\frac{1-u_i}{1-u_i t}\right) \prod_{j=1}^{J} (1-v_j)^{-1} \bigl(z_1^{\alpha_1} \cdots z_k^{\alpha_k}\bigr)$$

where $I = k^2 - k$, $J = \binom{k^2}{2}$, $f(t) = 1$, and where each u_i and v_j is of total degree 0 in the z and the sum of the α is equal to $-k$. Note that I_k has no poles at any $z_i = 0$. Inductively, we will prove that, for $a \geqslant 1$, each I_a will be a sum of integrals, each with respect to a variables over the torus $|z| = 1$, and each of the form:

$$f(t) \prod_{i=1}^{I} \left(\frac{1-u_i}{1-u_i t}\right) \prod_{j=1}^{J} (1-v_j)^{-1} \bigl(z_{i_1}^{\alpha_1} \cdots z_{i_a}{}^{\alpha_{i_a}}\bigr)$$

subject to the conditions

- each u_i and v_j is a monomial of total degree 0 in the z, possibly times a root of unity;
- $\alpha_{i_1} + \cdots + \alpha_{i_a} \geqslant -(k-a)$ and at least one z_i does not have a pole at 0;
- $f(t)$ is a (Laurent) polynomial in t, with zero at $t = 1$ of degree A, where $J - A \leqslant \binom{k^2}{2} - (k-a)$.

Each I_a will be constructed from I_{a+1} by replacing each integral by the sum of residues at some z_i, chosen not to have a poles at 0. In order to prove that I_a has the form claimed, we need to examine the residues of such terms at z_i, where z_i has no pole at 0. Say that there is a pole at $z_i = w$ of degree N. Note that the hypotheses that u_i and v_j have total degree zero imply that w is of degree one. We need to evaluate expressions of the form (14), where f has the above properties. By the product rule, we need to consider derivatives of four types of functions: $1/(1 - v z_i^{\alpha})$, where $1 - v w^{\alpha} \neq 0$; $(1 - u z_i^{\alpha})/(1 - u z_i^{\alpha} t)$, where $1 - u w^{\alpha} t \neq 0$; $(z_i - w)/(1 - (\frac{z_i}{w})^{\alpha})$; and $(z_i - w)(1 - (\frac{z_i}{w})^{\alpha} t^{-1})/(1 - (\frac{z_i}{w})^{\alpha})$. Define $T^N(f) = \lim_{z_i \to w} \frac{\partial^N f}{\partial z_i^N}$ for $N \geqslant 0$. We now break off the computation of the $T^N(f)$ as a sublemma.

Sublemma.

1. $T^N(1/(1-vz_i^\alpha)$ *where* $1-vw^\alpha \neq 0$ *is a sum of fractions with denominators* $(1-vw^\alpha)^s$, *where* $s \leqslant N+1$, *and with numerators of degree* $-N$.
2. $T^N((1-uz_i^\alpha)/(1-uz_i^\alpha t))$, *where* $1-uw^\alpha t \neq 0$ *is a sum of fractions with denominators* $(1-uw^\alpha)^s$, *where* $s \leqslant N+1$, *and with numerators of degree* $-N$. *Moreover, the term with* $s=N+1$.
3. $T^N((z_i-w)/(1-(\frac{z_i}{w})^\alpha))$ *is equal to* w^{1-N} *times a constant.*
4. $T^N((z_i-w)(1-(\frac{z_i}{w})^\alpha t^{-1})/(1-(\frac{z_i}{w})^\alpha))$ *is equal to* w^{1-N} *times a polynomial in* t^{-1}. *If* $u=0$ *then this polynomial is* $(1-t^{-1})$.

Proof. The first two cases follow from an easy induction argument which we leave to the reader. The third and forth have an elegant proof using the chain rule, which was supplied to us by M. Ash. For simplicity, consider only the third case, and let $y=\frac{z_i}{w}$. Then $f(z)=g(y)$, where $g(y)=w(1-y)/(1-y^\alpha)$. The chain rule now implies that $\mathrm{d}f/\mathrm{d}z=w^{-1}\mathrm{d}g/\mathrm{d}y$, and so $\mathrm{d}^N f/\mathrm{d}z^N=w^{-N}\mathrm{d}^N g/\mathrm{d}y^N$. □

When we multiply each term in I_a by some $(z_i-w)^N$ this decreases $J-A$ by N, by the $N=0$ cases of 3 and 4 in the sublemma. And, if we compare the effect of T^M with T^{M+1} in the four cases, we see that each additional derivative increases $J-A$ by at most 1.

Now we consider the change in the α_i. Again, by the $N=0$ cases of 3 and 4 in the sublemma, when we multiply each term in I_a by some $(z_i-w)^N$ we increase $\alpha_{i_1}+\cdots+\alpha_{i_a}$ by N; and, again, each derivative decreases it by at most 1. This implies that there will by some $\alpha_j \geqslant -1$. The corresponding z_j will not have a pole at 0, because of the presence of terms of the form $(1-z_h z_j^{-1})$ in the denominator. This proves the induction hypotheses.

Now consider I_1. We now have a sum of integrals, each with respect to only one of the z_i. But, since each u_i and v_j is of total degree zero, each will just be a root of 1 times a power of t. Hence, I_1 will be a sum of terms of the form

$$\oint_{|z|=1} f(t)\prod_{i=1}^{I}\left(\frac{1-\omega_i t^{n_i}}{1-\omega_i t^{n_i+1}}\right)\prod_{j=1}^{J}\left(1-\zeta_j t^{m_j}\right)^{-1} z^\alpha\,\mathrm{d}z,$$

where the ω_i and ζ_j are roots of 1. The integral will be zero unless $\alpha=1$, in which case the order of the pole at $t=1$ will be $J-A$. By induction this is at most $\binom{k^2}{2}-(k-1)$. Multiplying back in the factor of $(1-t)^{-k}$ gives the desired degree.

Theorem. *For each of* C, R, $\overline{C}$ *and* $\overline{R}$ *the colength series* $l(n)$ *satisfies* $l(n)=\alpha n^{\binom{k^2}{2}}+O(n^{\binom{k^2}{2}-1})$, *for some* α.

Proof. By Lemma 1, it suffices to do the case of $\overline{C}$ and $\overline{R}$. But, by Lemmas 8 and 9, $L(\overline{C})$ and $L(\overline{R},t)$ are rational functions, with unique highest order pole of order $\binom{k^2}{2}+1$ at 1. The lemma now follows from Eqs. (12) and (13). □

3. Examples and conjectures

In the case of 2×2 matrices it is possible to compute the generating functions for the colength sequences, although this is probably only of technical interest, since the multiplicities m_λ are known in this case, cf. [4]. At any rate, Lemma 7 implies that

$$L(\overline{C}, t) = \frac{1}{2(2\pi i)^2} \oint \frac{(1 - \frac{z_1}{z_2})(1 - \frac{z_2}{z_1})}{Q} \, \mathrm{d}\nu, \tag{15}$$

$$L(\overline{R}, t) = \frac{1}{2(2\pi i)^2} \oint \frac{(1 - \frac{z_1}{z_2})(1 - \frac{z_2}{z_1})(2 + \frac{z_1^2}{z_2^2} + \frac{z_2^2}{z_1^2})}{Q} \, \mathrm{d}\nu, \tag{16}$$

$$\text{where } Q = (1-t)^2\big(1-t^2\big)^2\left(1 - \frac{z_1}{z_2}t\right)\left(1 - \frac{z_2}{z_1}t\right)\left(1 - \frac{z_1}{z_2}t^2\right)^2\left(1 - \frac{z_2}{z_1}t^2\right)^2.$$

With a bit of help from the computer we were able to evaluate these integrals. Here are the results.

Example. The Poincaré series $L(\overline{C}, t)$ and $L(\overline{R}, t)$ for 2×2 matrices are given by

$$L(\overline{R}, t) = \frac{1 - t + t^2 - t^3 + t^4}{(1-t)^7(1+t)^3(1+t^2)^3(1+t+t^2)^2} = \frac{1 + t^5}{(1-t)(1-t^2)(1-t^3)^2(1-t^4)^3},$$

$$L(\overline{C}, t) = \frac{1}{(1-t)^7(1+t)^3(1+t^2)(1+t+t^2)^2} = \frac{1}{(1-t)^2(1-t^2)^2(1-t^3)^2(1-t^4)}.$$

We now conclude this paper with a series of conjectures. The techniques of Section 1 suggest the following. Let A be the algebra of $k \times k$ matrices with involution, either symplectic or orthogonal. Let $U(A)$ be the universal algebra for A on k^2 generators. The transcendence degree of the center was calculated in [3]. One can generalize all of the material from Section 1 to this case and get lower bounds for the colength sequences. Our first conjectures are that these give the asymptotic behavior.

Conjecture 1. *Let A be the algebra of $k \times k$ matrices with symplectic involution, where k is even. Then the colength sequences of A is $\alpha n^K + O(n^{K-1})$, where $K = \binom{k^2}{2} + \binom{k}{2} - 1$.*

Conjecture 2. *Let A be the algebra of $k \times k$ matrices with orthogonal involution. Then the colength sequences of A is $\alpha n^K + O(n^{K-1})$, where $K = \binom{k^2}{2} + \binom{k+1}{2} - 1$.*

Let $\Lambda_h(n)$ be the partitions of n of height at most h. Given a cocharacter sequence $\{\chi_n(A)\}$ with each $\chi_n(A) = \sum m_\lambda(A)\chi^\lambda$, we can define the height h colength sequence to be $l(n,h)(A) = \sum_{\lambda \in \Lambda_h(n)} m_\lambda$. Knowing the asymptotic behavior of these bounded height colength sequences might help to formulate a conjecture about the asymptotic behavior of the multiplicities themselves. At any rate, lower bounds and conjectures follow from the transcendence degree of the center of the universal algebra in h generators, just as above. Here are some conjectures:

Conjecture 3. *Let A be the algebra of $k \times k$ matrices. Then for each $k^2 \geqslant h \geqslant 2$, $l(n,h)(A)$ is $\alpha n^K + O(n^{K-1})$ where $K = k^2(h-1) - \binom{h-1}{2}$.*

Conjecture 4. *Let A be the algebra of $k \times k$ matrices with involution. Then for each $k^2 \geqslant h \geqslant 2$, $l(n,h)(A)$ is $\alpha n^K + O(n^{K-1})$ where $K = k^2(h-1) + \binom{k}{2} - \binom{h-1}{2}$ in the symplectic case and $K = k^2(h-1) + \binom{k+1}{2} - \binom{h-1}{2}$ in the orthogonal case.*

References

[1] A. Berele, Homogeneous polynomial identities, Israel J. Math. 42 (1982) 258–272.
[2] A. Berele, A. Regev, Some remarks on trace cocharacters, J. Algebra 176 (1995) 1013–1024.
[3] A. Berele, D.J. Saltman, The centers of generic division algebras with involution, Israel J. Math 63 (1988) 98–118.
[4] V.S. Drensky, Codimensions of T-ideals and Hilbert series of relatively free algebras, J. Algebra 91 (1984) 1–17.
[5] E. Formanek, Invariants and the ring of generic matrices, J. Algebra 89 (1984) 178–223.
[6] E. Formanek, A conjecture of Regev about the Capelli polynomial, J. Algebra 109 (1987) 93–114.
[7] I.N. Herstein, Notes from a Ring Theory Conference, CBMS, vol. 9, Amer. Math. Soc., 1971.
[8] G. James, A. Kerber, The Representation Theory of the Symmetric Group, Addison–Wesley, Reading, MA, 1981.
[9] I.G. MacDonald, Symmetric Functions and Hall Polynomials, second ed., Oxford University Press (Clarendon), 1995.